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LETTER TO THE EDITOR

Intermittency as multifractality in history space

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Abstract. Temporal intermittency in a chaotic dynamical system can be regarded as a manifestation of the multifractal properties of the set of histories. The Renyi entropies K_q which generalise the Kolmogorov entropy K_1 play a role analogous to the exponents $\tau(q)$ characterising the scaling properties of the moments of the mass distribution. We introduce the topological entropy $h(\lambda)$ of the subset of histories with the same local expansion parameter. We show that K_q and $h(\lambda)$ are related to each other by means of a Legendre transformation.

In the characterisation of the attractor in dynamical systems a hierarchy of exponents $\tau(q)$ has been introduced, describing the scaling behaviour of the moments of its mass density with respect to phase space dilations (Grassberger 1983, Badii and Politi 1984, Paladin and Vulpiani 1984).

The structure of the attractor can be described as an interwoven family of singularities of type α, where α is defined by the behaviour $p(l) \propto l^{\alpha}$ of the probability of finding a point of the attractor within a distance l of a singularity of type α. These singularities are distributed over a set of fractal dimension $f(\alpha)$. The exponents $\tau(q)$ and the dimensions $f(\alpha)$ are related to each other by means of a Legendre transformation (Frisch and Parisi 1983, Benzi *et al* 1984, Jensen *et al* 1985, Halsey *et al* 1986).

In this description all information about the temporal evolution of a given point in phase space is lost, although the probability $p(l)$ is sometimes estimated by computing the first return time $T(l)$ within a distance l of a given point, and by setting $p(l) \sim (T(l))^{-1}$.

We wish to point out that further information can be retrieved by considering the dilation of an element of phase space under time translations. The natural setting of this analysis is the set of all histories.

The role corresponding to the exponents $\tau(q)$ is played by the Renyi entropies K_q (Renyi 1970), themselves a generalisation of the Kolmogorov entropy K_1. We introduce a parameter λ characterising the local divergence of a volume in the tangent space, and the topological entropy $S(\lambda)$ of trajectories with a given value of λ. The Renyi entropies K_q and the topological entropy $S(\lambda)$ are then related to each other by a Legendre transformation analogous to that relating $\tau(q)$ and $f(\alpha)$ to each other.

We now briefly recall the concepts allowing for a description of strange attractors as multifractal objects in phase space (Frisch and Parisi 1983, Benzi *et al* 1984, Jensen *et al* 1985, Halsey *et al* 1986). Let $d\mu(x)$ be the normalised invariant measure on the strange attractor. Let x_0 be a point of the attractor. We say that x_0 is characterised by

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a singularity of type α if the probability $p(l)$ of finding a point of the attractor in a ball of radius l around x_0 scales like l^α when $l \to 0$:

$$p_l(x_0) = \int_{|x-x_0|<l} \mathrm{d}\mu(x) \sim l^\alpha. \tag{1}$$

The type of the singularity will in general depend on the point one is looking at. A way of describing the distribution of singularities is to divide the portion of phase space occupied by the attractor into boxes of linear size l, and to compute for each box i the probability $p_i(l)$ that a point (sampled according to the invariant measure $\mathrm{d}\mu(x)$) belongs to the ith box. We can then compute the moments $\gamma(q, l)$ of the distribution of the $p_l(i)$:

$$\gamma(q, l) = \sum_{\substack{\text{boxes}\\ i}} p_l(i)^q \sim l^{\tau(q)}. \tag{2}$$

If the attractor were a homogeneous fractal of fractal dimension D one would have

$$\tau(q) = D(q-1). \tag{3}$$

In general, however, the exponents $\tau(q)$ deviate from the linear law (3). We say in this case that the attractor is a multifractal and we consider a whole family of dimensions D_q (the Renyi dimensions) defined by

$$\tau(q) = D_q(q-1). \tag{4}$$

The space average appearing in (2) can be estimated by means of an average taken over the singularity distribution. Let us collect the points x with a singularity belonging to the interval $[\alpha, \alpha + \mathrm{d}\alpha]$ into a subset $\Omega(\alpha)$. The fractal dimension of $\Omega(\alpha)$ is $f(\alpha)$ in the sense that if the phase space is partitioned into boxes of linear size l, the number of boxes necessary to cover $\Omega(\alpha)$ increases, as $l \to 0$, as

$$\mathcal{N}_\alpha \sim l^{-f(\alpha)}. \tag{5}$$

The coefficient of this proportionality law can be assumed to be a smoothly varying measure $\mathrm{d}\nu(\alpha)$. By grouping together boxes dominated by the same singularity α, we then have

$$\sum_i (p_l(i))^q = \int \mathrm{d}\nu\,(\alpha) l^{-f(\alpha)} l^{\alpha q}. \tag{6}$$

By a saddle-point estimate of this integral we have in the limit $l \to 0$

$$\tau(q) = \min_\alpha [\alpha q - f(\alpha)]. \tag{7}$$

The fractal dimension D_0 is given by $D_0 = -\tau(0)$ whereas the dimension $D_1 = \lim_{q\to 1} \tau(q)/(q-1)$ is the information dimension. At $q=0$ one picks up the most probable singularity α^* where $f(\alpha)$ attains its maximum value, $f(\alpha^*) = D_0$, while at $q = 1$ the average value of α given by the information dimension $D_1 \leqslant D_0$ where $f(\alpha = D_1) = D_1$.

Let us now show that the analysis of the local structure of the set of histories of a dynamical system can be carried out in close analogy with the formalism described above. Let us consider a dynamical system with F degrees of freedom evolving in continuous time according to the law

$$\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{f}(\boldsymbol{x}) \qquad \boldsymbol{x}, \boldsymbol{f} \in \mathbb{R}^F. \tag{8}$$

We consider a record of the history of the system, obtained by identifying the state $\boldsymbol{x}(t)$ of the system at regular time intervals of length τ:

$$\boldsymbol{x}_k = \boldsymbol{x}(k\tau) \qquad k \in \mathbb{Z}. \tag{9}$$

A *history* $\boldsymbol{X}^{(N)}$ of the system is a finite subset of N consecutive $\boldsymbol{x}_k$:

$$\boldsymbol{X}_k^{(N)} = (\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_{k+N-1}) \tag{10}$$

with $\boldsymbol{X}^{(N)} \in \mathbb{R}^{FN}$. We can introduce a metric in the space of histories by considering, e.g.,

$$d(\boldsymbol{X}_k, \boldsymbol{Y}_l) = \sup_{1 \leq j \leq N} |\boldsymbol{x}_{k+j} - \boldsymbol{y}_{l+j}|. \tag{11}$$

We can fix our attention on one history $\boldsymbol{X}_0^{(N)}$ and on the l-ball $B_l(\boldsymbol{X}_0)$ around it:

$$B_l(\boldsymbol{X}_0^{(N)}) = \{\mathbf{X}^{(N)}: d(\boldsymbol{X}^{(N)} - \boldsymbol{X}_0^{(N)}) \leq l\}. \tag{12}$$

A natural measure in the space of histories is introduced if the initial state $\boldsymbol{x}_k$ is distributed according to the invariant measure on the attractor. We can thus define the probability $P_l(\boldsymbol{X}_0^{(N)})$ that a given history falls within the l-ball $B_l(\boldsymbol{X}_0^{(N)})$ around $\boldsymbol{X}_0^{(N)}$. Such a probability can be estimated (Grassberger and Procaccia 1983) if we know a great number $\mathcal{N}$ of histories $\boldsymbol{X}_j^{(N)}$ (or a very long record of consecutive states of the system, which is then broken up into histories of length N)

$$P_l(\boldsymbol{X}_0^{(N)}) \simeq \frac{1}{\mathcal{N}} \sum_j \theta[l - d(\boldsymbol{X}_0^{(N)} - \boldsymbol{X}_j^{(N)})]. \tag{13}$$

If N is increased and l is reduced we expect $P_l(X_0)$ to scale like

$$P_l(\boldsymbol{X}_0^{(N)}) \sim l^\beta \exp(-\lambda\tau N) \tag{14}$$

where β and λ depend in general on $\boldsymbol{X}_0^{(N)}$. Our aim is to characterise the distribution of λ (a sort of local expansion parameter) much in the same way as the distribution of α for the phase space has been characterised.

It is convenient at this point to divide the space of histories into boxes. This can be done by partitioning the phase space into boxes of linear size l. The 'history' boxes are then the cartesian product (for $1 < j < N$) of the 'phase space' boxes. The time evolution of the system is thus given by a particular history box $I = (i_1, \ldots, i_N)$ where $i_1, \ldots, i_N$ are the indices of the 'phase space' boxes visited at times $(t+\tau, t+2\tau, \ldots, t+N\tau)$. We can now introduce the probability $P_l(I)$ that a history belongs to the history box I. This is of course analogous to $p_l(i)$. The moments $\Gamma_{N,l}(q)$ of $P_l(I)$ are given by

$$\Gamma_{N,l}(q) = \sum_I (P_l(I))^q. \tag{15}$$

We expect therefore

$$\Gamma_{N,l}(q) \sim \exp(-K_q(q-1)\tau N). \tag{16}$$

The exponents K_q, heuristically defined in (16), are the Renyi entropies (Renyi 1970) and generalise the Kolmogorov entropy K_1. Namely, one has

$$K_q = -\lim_{\tau\to 0} \lim_{l \to 0} \lim_{N\to\infty} \frac{1}{\tau N(q-1)} \ln\left(\sum_I (P_l(I))^q\right). \tag{17}$$

One can show that $K_q \leq K_{q'}$ if $q > q'$. The number $\mathcal{N}_N$ of boxes necessary to cover the set of histories in $\mathbb{R}^{FN}$ is given by

$$\mathcal{N}_N \sim \exp(K_0\tau N) \qquad \text{for } N \to \infty. \tag{18}$$

We can thus identify K_0 with the topological entropy S. The Renyi entropies allow therefore for a characterisation of the intermittency of a chaotic signal (Paladin and Vulpiani 1986).

The exponential decay of $P_l(I)$ (equation (14)) is related to the local divergence of nearby orbits, which takes place also if the motion is globally confined. We can thus introduce a local expansion parameter (LEP) λ to measure the rate of divergence of an F-dimensional volume in tangent space:

$$P_l(I) \sim \exp(-\lambda \tau N) \qquad \text{for large } N. \tag{19}$$

We now consider the set $\Omega(\lambda)$ of histories whose LEP belongs to the interval $[\lambda, \lambda + \mathrm{d}\lambda]$. The number $\mathcal{N}_\lambda$ of boxes necessary to cover this set will increase with N according to the law

$$\mathcal{N}_\lambda \sim \exp(\tau N h(\lambda)). \tag{20}$$

We can consider $h(\lambda)$ as the topological entropy $S(\lambda)$ of the histories belonging to $\Omega(\lambda)$, with the warning that if $h(\lambda) \leqslant 0$, then $S(\lambda) = 0$. We have by definition $h(\lambda) \leqslant K_0$.

We now show that $h(\lambda)$ and K_q are related to each other by a Legendre transformation analogous to (7).

When we consider the moments $\Gamma_{N,l}(q)$ in (15) we can group together all boxes with a given value of λ and obtain, in analogy with (6),

$$\Gamma_{N,l}(q) \sim \int \mathrm{d}\rho(\lambda) \exp(\tau N h(\lambda)) \exp(-\tau N \lambda q)$$

where $\mathrm{d}\rho(\lambda)$ is a smooth measure.

Estimating this integral by the saddle-point method in the limit $N \to \infty$ yields

$$\Gamma_{N,l}(q) \sim \exp[-\tau N K_q (q-1)] \tag{21}$$

with

$$K_q = \frac{1}{(q-1)} \min[q\lambda - h(\lambda)]. \tag{22}$$

We see that $\mathrm{e}^{-\tau N}$ plays in (20) a role analogous to l in (6). Equation (22) is strictly analogous to (7). Of course λ is bounded between the extrema

$$\lambda_{\min} = K_\infty \qquad \lambda_{\max} = K_{-\infty}. \tag{23}$$

If $h(\lambda)$ is differentiable, a value $\bar{\lambda}$ is picked up by (22) satisfying

$$q = \left.\frac{\mathrm{d}h}{\mathrm{d}\lambda}\right|_{\lambda=\bar{\lambda}}. \tag{24}$$

At $q = 0$, $\bar{\lambda} = K^*$, whereas $h(\lambda)$ attains its maximum value:

$$h(\lambda = K^*) = K_0. \tag{25}$$

It is easy to show that $h(\lambda)$ is a convex function defined on the interval $[K_\infty, K_{-\infty}]$.

Around its maximum, $h(\lambda)$ can be approximated by a parabola:

$$h(\lambda) = K_0 - (1/2\mu)(\lambda - K^*)^2. \tag{26}$$

We have, for small enough q,

$$qK_{q+1} = qK_1 - \tfrac{1}{2}\mu q^2. \tag{27}$$

If the $P_l(I)$ are distributed according to a log-normal law, (27) would remain true for all values of q

$$\mu = \lim_{\tau\to 0}\lim_{l\to 0}\lim_{N\to\infty}\left(\frac{1}{\tau N}[\langle \ln^2 P_l(I)\rangle - \langle \ln P_l(I)\rangle^2]\right) \tag{28}$$

where $\langle\ \rangle$ means average on the boxes I.

Let us also remark that the relation between $h(\lambda)$ and K_q is reminiscent of that linking the entropy to the free energy in thermodynamics. The role of temperature is played by the inverse moment index $T = q^{-1}$. The 'free energy' $K_q(q-1)$ is linear in T only when there is no intermittency (i.e. the LEP does not fluctuate).

On the other hand, as $T\to 0^+$ (i.e. as $q\to\infty$) the system is found in the state of minimal energy $\lambda_{\min}$ and of minimal entropy $h(\lambda_{\min})$.

One may also speculate on the possible existence of 'phase transitions' which would appear as edges in the K_q against q curve (Coniglio 1986). We finally want to recall that the Renyi entropies can be extracted by a numerical calculation (Paladin and Vulpiani 1986) of

$$\overline{n^q(l)} \equiv \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}(n_m(l))^q \approx \langle P_l(I)^{q+1}\rangle \tag{29}$$

where M is the number of records $X_m^{(N)}$ available and

$$n_i(l) = \frac{1}{(M-1)}\sum_{j\neq i}\theta(l - |X_i^{(N)} - X_j^{(N)}|).$$

Let us note that we are using a Euclidean metric here which is not the sup metric of (11). However in a finite-dimensional space all metrics are equivalent. Moreover it was shown (Paladin and Vulpiani 1986) that in many cases the Renyi entropies are related to the generalised Lyapounov exponents $L^{(p)}(q)$ (Benzi *et al* 1985, Paladin and Vulpiani 1986):

$$K_{q+1} = L^{(p)}(-q)/-q. \tag{30}$$

Here p is the number of non-negative Lyapounov exponents γ_i of the systems and $\lim_{q\to 0} L^{(i)}(q)/q = \Sigma_{k=1}^{i}\gamma_k$ (with $\gamma_k \geq \gamma_{k+1}$). One then sees that (30) becomes the Pesin relation (Pesin 1976) $K_1 = \Sigma_{k=1}^{p}\gamma_k$ in the limit of vanishing q.

We have shown, in summary, that the Renyi entropies K_q can be put in relations with the distribution of local expansion parameters λ in the space of histories by a thermodynamic formalism quite close in spirit to that introduced to describe the multifractal structure of the attractors in phase space.

After the completion of this paper we discovered that Eckmann and Procaccia (1986) obtained results similar to ours.

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